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ABSTRACT

A MACSYMA program is presented which determines whether a given single nonlinear ODE or PDE with (real) polynomial terms fulfills the necessary conditions for having the Painlevé property. Together with some mathematical background, we give a synopsis of the algorithm for the program, its scope and limitations. Various examples of typical output of the program are provided.

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This work is partially supported by AFOSR under Grant 85-NM-0263. We thank Jonathan Len (Symbolics, Inc.) for writing the POWERS subroutine. The original co-developer of the program, Dr. Eric Van den Bulck, presently at the University of Leuven (Belgium), is gratefully acknowledged for endless nights of debugging. Help from Dr. D. Rand (University of Montréal) in further debugging of the program was very much appreciated.

THE PAINLEVÉ TEST FOR NONLINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Willy Hereman and Sigurd Angenent [1]

1 Introduction: Mathematical Background

Roughly speaking dynamical systems may be divided into two classes. In the first class one finds those systems which exhibit chaotic behavior, i.e. the solutions of such systems depend sensitively on the initial data. A consequence of this chaotic behaviour is that such systems are usually not explicitely solvable in terms of "elementary functions." The other class of systems contains those equations which are *algebraically completely integrable.* Obviously, one wants to characterize systems which do not lead to chaos.

At the turn of the century, Painlevé and his collaborators were able to identify all second order ODEs of the form $f_{xx} = K(x, f, f_x)$, which are globally integrable in terms of elementary functions by quadratures or by linearization. The restrictions on the function K, which is rational in f_x, algebraic in f, and analytic in x, arise from careful singularity analysis. Indeed, integrability requires that the only *movable* singularities in the solution $f(x)$ are *poles.* Singularities are *movable* if their location depends on the initial conditions. Hence, the critical points (including logarithmic branch points and essential singularities) ought to be fixed to have integrability and consequently to assure predictable behavior of solutions for all initial conditions.

Definition: A simple equation or system is said to have the *Painlevé property (PP)* if its solution in the complex plane has no worse singularities than movable poles.

In a broader, more modern context, including PDEs, integrability became associated with the existence of a Lax representation which allows *linearization* of the given equation(s). The solution may then be constructed by e.g. the Inverse Scattering Transform.

A restricted class of completely integrable PDEs, admits solitary traveling wave solutions (called *solitons* if they conserve their identity upon collision). Completely integrable systems of ODEs admit a finite dimensional Hamiltonian formulation and have a finite number of first integrals. Analogously, completely integrable PDEs possess an infinite number of conserved quantities, infinitely many symmetries, nontrivial prolongation structures and Kac-Moody algebras.

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In a first attempt to identify integrable PDEs, Ablowitz *et al.* [for references: consult the books and review papers cited at the end] conjectured that every ODE, obtained by an exact reduction of a PDE solvable by IST, possesses the Painlevé property. This conjecture was based on the fact that well-known integrable PDEs, such as the Korteweg-de Vries (KdV) equation, can be transformed (by similarity transformations) into ODEs of Painlevé type, in particular into the so-called six Painlevé transcendents. This is yet another indication that integrability is closely related to the absence of movable critical points.

2 Algorithm

An algorithm, due to Weiss *et al* enables us to verify if the given ODE or PDE satisfies the necessary criteria to have the Painlevé property.

For the PDE case: The solution f, say in only two independent variables (t, x), expressed as a Laurent series,

$$f = g^{\alpha} \sum_{k=0}^{\infty} u_k g^k \tag{1}$$

should only have movable poles. In (1), $u_0(t,x) \neq 0$, α a negative integer, and $u_k(t,x)$ are analytic functions in a neighborhood of the singular; non-characteristic manifold $g(t,x) = 0$, with $g_x(t,x) \neq 0$.

For the ODE case[2]: x will be replaced by $g + x_0$ in (1); x_0 being the initial value for x.

The Painlevé test is carried out in three steps:

- **Step 1**: Determine the negative integer α and u_0 from the leading order "ansatz" $f \propto u_0 g^{\alpha}$, by balancing the minimal power terms after substitution of this first term of (1) into the given equation
- **Step 2**: Calculate the non-negative integer powers r, called the *resonances*, at which arbitrary functions u_r enter the expansion. This is done by requiring that u_r is arbitrary after substitution of $f \propto u_0 g^{\alpha} + u_r g^{\alpha+r}$ into the equation, only retaining its most singular terms
- **Step 3**: Verify that the correct number of arbitrary functions u_r indeed exists by substituting a truncated expansion of the form (1), in which one sums over $k = 1, 2, ..., rmax$, where $rmax$ represents the largest resonance, into the given equation.

  At non-resonance levels, determine u_k unambiguously.

  At resonance levels, u_r should be arbitrary due to a vanishing coefficient of $g^{r+minpowg}$. One has to check whether or not this *compatibility condition*

[2]For this case a MACSYMA package already existed [Ran]

is satisfied. Here *minpowg* denotes the (negative) power in g of the most singular terms in the equation.

An equation or system for which the above steps can be carried out consistently, and for which the compatibility conditions at all resonances are satisfied, is said to have the Painlevé Property and is conjectured to be integrable.

The reader should be warned that the above algorithm does not detect the existence of essential singularities. In other words, for an equation to be integrable it is **necessary** but **not** yet **sufficient** that it passes the Painlevé test. There are integrable equations (see Ex. 2) that only have the Painlevé property after a suitable chance of variables.

Refinements of the Painlevé property have been established, allowing for rational power expansions of f, and hence including certain algebraic branch points in addition to movable poles.

3 Scope and Limitations of the Program

3.1 Scope

- the program works for a single ODE or PDE
- the degree of nonlinearity in all the variables is unlimited
- the number of parameters in the equation is unlimited
- the number of independent variables is also unlimited[3]
- ODEs and PDEs may have explicitly given space (and/or time) dependent coefficients of integer degree (see Ex. 4)
- PDEs may have arbitrary time and space dependent coefficients (see Ex. 4)
- coefficients may be complex, although the usefulness of the Painlevé test is then debatable
- a selected positive or negative rational value of α, or $\alpha = 0$ can be supplied by the user
- the time consuming calculation of the coefficients u_k and the verification of the compatibility conditions is made optional
- it is possible to substitute an expansion of the form (1) with a selected number of terms, e.g. to carry on with the calculations beyond *rmax*

[3] *The released version of the program works for at most four variables (t, x, y, z) but this can be extended trivially*

- the output provides vital information, including error messages and warnings, to remedy possible problems

3.2 Limitations

- systems of equations are presently excluded
- the test is restricted to the traditional Painlevé test based on the expansion (1), with at least rational α, hence general fractional expansions in g are at present not possible
- transcendental terms in the equation are not allowed, but they can often be removed by a suitable (exponential) transformation of the dependent variable (see Ex. 2)
- arbitrary parameters in the powers of f and its derivatives are not allowed
- neither are arbitrary (unspecified) functions of f and its derivatives
- selective substitution of certain u_k is presently not possible, i.e. u_0, u_1, etc. are explicitly determined whenever possible, and their expressions are used in the calculation of the next u_k
- apart from trivial cases, nonlinear equations for u_0 are not solved; if they occur the program carries on with the undetermined coefficient u_0 (see Ex. 5)
- the program only checks whether or not the compatibility condition is satisfied. It does not solve for arbitrary parameters (or functions) or for u_0 and its derivatives, should these occur (see Exs. 3, 4 and 5)
- intermediate output is only possible by including extra *print* statements in the source code of the program
- the expressions occuring in the output on the screen are not accessible for further interactive calculations without modification of the program

4 Spin-offs of the Painlevé Analysis

The Painlevé test, apart from it usefulness in testing the integrability of Hamiltonian systems, evolution and wave equations, has rather interesting connections with standard techniques in the study of dynamical systems and soliton theory:

- Truncation of the Laurent series (1) at the constant level term leads to auto-Bäcklund transformations
- The resulting Painlevé-Bäcklund equations, obtained by substitution of the truncated expansion and equating to zero powers of g, can readily be linearized to derive Lax pairs for various systems of ODEs and PDEs
- As a consequence for ODEs, it is possible to construct algebraic curves and explicitly integrate the equations of motion
- For PDEs, Painlevé analysis provides insight in the construction of single and multi-soliton solutions and also rational solutions, via direct methods (Hirota's formalism and its clones) (see Exs. 3 and 5)
- The Painlevé property serves as in tool in identifying the infinite dimensional symmetry algebras for PDEs, which in turn have the structure of subalgebras of Kac-Moody-Virasoro algebras

5 Using the Program

The program carries out the Painlevé test in batch mode without interaction by the user. The user only has to type in the LHS of the equation (denoted by eq) and possibly select some options.

- For ODEs: The use of dependent variable f and independent variable x is mandatory. A typical term in the ODE reads $fx[.](x)$, where within the brackets the order of derivation is inserted. The function without derivatives may be denoted by f itself. The symbol eq denotes the LHS of the equation
- Ex.: To test the Fisher ODE, $f_{xx} + af_x - f^2 + f = 0$, one would enter

  $$eq : fx[2](x) + a * f[x](x) + f * * 2 - f;$$

  The program will then treat a as an arbitrary parameter.
- For PDEs: Analoguously, a typical term reads $ftxyz[k, l, m, n](t, x, y, z)$, where the integers k, l, m, and n are the orders of derivation with respect to the variables t, x, y, and z

- Ex.: To test the KdV equation, $f_t + aff_x + f_{xxx} = 0$, one enters

$$eq : ftx[1,0](t,x) + a * f * ftx[0,1](t,x) + ftx[0,3](t,x);$$

  Again, the program will treat a as an arbitrary parameter.

The program is available (on IBM floppy or Macintosh disk) for two slightly different version of MACSYMA: Franz Lisp MACSYMA 309 and Common Lisp MACSYMA 412. Free copies may be obtained at the above address. To get the program by email, contact: HEREMAN@EUNICE.MRC.WISC.EDU (arpa net).

The program has been successfully tested, with MACSYMA versions 309.3 and 412, on a VAX 2000 and a VAX 11/780, for more than 100 ODEs and PDEs.

6 Examples

The selected examples reflect some of the capabilities of the program. In the five examples, a, b and c are arbitrary parameters, and a(t) is an arbitrary function. Many more examples, tested with this program, may be found in the cited books [Lea, Ste].

Example 1: The Korteweg-de Vries Equation

For the ubiquitous KdV equation,

$$f_t + ff_x + bf_{xxx} = 0, \tag{2}$$

the program provides the following output:

PAINLEVE ANALYSIS OF EQUATION, $bf_{xxx} + ff_x + ft = 0$

SUBSTITUTE $u_0\ g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE $[2\ alfa - 1, alfa - 3]$

* COEFFICIENT OF $g^{2\ alfa-1}$ IS $u_0{}^2\ alfa\ g_x$

* COEFFICIENT OF g^{alfa-3} IS $u_0\ (alfa-2)\ (alfa-1)\ alfa\ b\ (g_x)^3$

FOR EXPONENTS $(2\ alfa - 1)$ AND $(alfa - 3)$ OF g, DO

WITH $alfa = -2$, POWER OF g is $-5 \longrightarrow$ SOLVE FOR u_0

TERM $-2\, u_0\, g_x\, (12\, b\, (g_x)^2 + u_0)\, \frac{1}{g^5}$ IS DOMINANT IN EQUATION.

WITH $u_0 = -12\, b\, (g_x)^2 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\, g^{alfa} + u_r\, g^{r+alfa}$ FOR f IN EQUATION

TERM $b\, (g_x)^3\, (r-6)\, (r-4)\, (r+1)\, u_r\, g^{r-5}$ IS DOMINANT IN EQUATION.

THE 2 NON-NEGATIVE INTEGRAL ROOTS ARE $[r = 4,\ r = 6]$

WITH MAXIMUM RESONANCE = $6 \longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{6} g^{k-2}\, u_k$ FOR f IN EQUATION.

WITH $u_0 = -12\, b\, (g_x)^2$

* COEFFICIENT OF $\frac{1}{g^4}$ IS $6b(g_x)^2((-12b(g_x)^2)_x - 36bg_xg_{xx} + 5u_1g_x)$

  $u_1 = 12bg_{xx}$

* COEFFICIENT OF $\frac{1}{g^3}$ IS $24bg_x(4bg_xg_{xxx} - 3b(g_{xx})^2 + u_2(g_x)^2 + g_tg_x)$

  $$u_2 = -\frac{4bg_xg_{xxx} - 3b(g_{xx})^2 + g_tg_x}{(g_x)^2}$$

* COEFFICIENT OF $\frac{1}{g^2}$ IS

  $$-12b(b(g_x)^2g_{xxxx} - 4bg_xg_{xx}g_{xxx} + 3b(g_{xx})^3 - g_tg_xg_{xx}$$
  $$-u_3(g_x)^4 + g_{tx}(g_x)^2)/g_x$$

  $$u_3 = \frac{b(g_x)^2g_{xxxx} - 4bg_xg_{xx}g_{xxx} + 3b(g_{xx})^3 - g_tg_xg_{xx} + g_{tx}(g_x)^2}{(g_x)^4}$$

* COEFFICIENT OF $\frac{1}{g}$ IS 0

  u_4 IS ARBITRARY !

  COMPATIBILITY CONDITION IS SATISFIED !

* COEFFICIENT OF 1 IS

  $$-(b^2(g_x)^4g_{xxxxxxx} - 9b^2(g_x)^3g_{xx}g_{xxxxxx} - 17b^2(g_x)^3g_{xxx}g_{xxxxx}$$
  $$+48b^2(g_x)^2(g_{xx})^2g_{xxxxx} - 2bg_t(g_x)^3g_{xxxxx} + 70b^2(g_x)^2g_{xx}(g_{xxx})^2$$
  $$-174b^2g_x(g_{xx})^3g_{xxx} + 17bg_t(g_x)^2g_{xx}g_{xxx} - 8bg_{tx}(g_x)^3g_{xxx}$$
  $$+81b^2(g_{xx})^5 - 21bg_tg_x(g_{xx})^3 + 21bg_{tx}(g_x)^2(g_{xx})^2$$
  $$+6u_4b(g_x)^6g_{xx} - 9bg_{txx}(g_x)^3g_{xx} + (g_t)^2(g_x)^2g_{xx} + 6u_5b(g_x)^8$$
  $$+6(u_4)_xb(g_x)^7 + g_{tt}(g_x)^4 + 2bg_{txxx}(g_x)^4 - 2g_tg_{tx}(g_x)^3)/(g_x)^5$$

$$
\begin{aligned}
u_5 = \quad & -(b^2(g_x)^4 g_{xxxxxxx} - 9b^2(g_x)^3 g_{xx} g_{xxxxxx} - 17b^2(g_x)^3 g_{xxx} g_{xxxxx} \\
& +48b^2(g_x)^2(g_{xx})^2 g_{xxxxx} - 2bg_t(g_x)^3 g_{xxxxx} + 70b^2(g_x)^2 g_{xx}(g_{xxx})^2 \\
& -174b^2 g_x(g_{xx})^3 g_{xxx} + 17bg_t(g_x)^2 g_{xx} g_{xxx} - 8bg_{tx}(g_x)^3 g_{xxx} \\
& +81b^2(g_{xx})^5 - 21bg_t g_x(g_{xx})^3 + 21bg_{tx}(g_x)^2(g_{xx})^2 + 6u_4 b(g_x)^6 g_{xx} \\
& -9bg_{txx}(g_x)^3 g_{xx} + (g_t)^2(g_x)^2 g_{xx} + 6(u_4)_x b(g_x)^7 + g_{tt}(g_x)^4 \\
& +2bg_{txxx}(g_x)^4 - 2g_t g_{tx}(g_x)^3)/(6b(g_x)^8)
\end{aligned}
$$

* COEFFICIENT OF g IS 0

  u_6 IS ARBITRARY !

  COMPATIBILITY CONDITION IS SATISFIED !

Example 2: The sine-Gordon Equation

The transcendental term in the *sine-Gordon* equation, in light cone coordinates,

$$u_{tx} - sin(u) = 0, \tag{3}$$

can be removed by the simple substitution $f = exp(iu)$ to obtain an equivalent equation with polynomial terms:

$$-2f_t f_x + 2ff_{tx} - f^3 + f = 0. \tag{4}$$

PAINLEVE ANALYSIS OF EQUATION, $-2f_t f_x + 2ff_{tx} - f^3 + f = 0$

SUBSTITUTE $u_0\, g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE $[2alfa - 2,\ 3\ alfa,\ alfa]$

* COEFFICIENT OF $g^{2\ alfa - 2}$ IS $-2\ u_0{}^2\ alfa\ g_t\ g_x$
* COEFFICIENT OF $g^{3\ alfa}$ IS $-u_0{}^3$
* COEFFICIENT OF g^{alfa} IS u_0

FOR EXPONENTS $2\ alfa - 2$ AND $3\ alfa$ OF g, DO

WITH $alfa = -2$, POWER OF g is 6 $\longrightarrow$ SOLVE FOR u_0

TERM $u_0{}^2\ (4g_t g_x - u_0)\frac{1}{g^6}$ IS DOMINANT IN EQUATION.

WITH $u_0 = 4g_t g_x \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\ g^{alfa} + u_r\ g^{r+alfa}$ FOR f IN EQUATION

TERM $8(g_t)^2(g_x)^2(r-2)(r+1)\, u_r\, g^{r-6}$ IS DOMINANT
IN EQUATION.
THE ONLY NON-NEGATIVE INTEGRAL ROOT IS $[r = 2]$
WITH MAXIMUM RESONANCE = 2 $\longrightarrow$ CHECK RESONANCES.
SUBSTITUTE POWER SERIES $\sum_{k=0}^{2} g^{k-2} u_k$ FOR f IN EQUATION.

WITH $u_0 = 4 g_t g_x$

* COEFFICIENT OF $\frac{1}{g^3}$ IS $-16(g_t)^2(4g_{tx} + u_1)(g_x)^2$
  $u_1 = -4\, g_{tx}$

* COEFFICIENT OF $\frac{1}{g^4}$ IS 0

  u_2 IS ARBITRARY !

  COMPATIBILITY CONDITION IS SATISFIED !

FOR EXPONENTS ($2\, alfa - 2$) AND ($alfa$) OF g, $alfa = 2$ IS NON-NEGATIVE.

FOR EXPONENTS ($3\, alfa$) AND ($alfa$) OF g, $alfa = 0$ IS NON-NEGATIVE.

Example 3: The Fisher Equation

¿From rigorous analysis it follows that if the initial datum is given by $u(0,x) = 1$ $(x \leq 0)$, $u(0,x) = 0$ $(x > 0)$, then the solution of the *Fisher* equation,

$$u_t - u_{xx} + u^2 - u = 0, \tag{5}$$

will converge to a travelling wave of speed $c = 2$. Furthermore, for every speed $c \geq 2$ there is a travelling wave with $u(t, -\infty) = 1, u(t, \infty) = 0$. In 1979, an exact closed form solution of (5) was constructed: $u(t,x) = U(x - x_0 - \frac{5t}{\sqrt{6}})$, where

$$U(\xi) = \frac{1}{4}\left(1 - \tanh\left(\frac{\xi}{2\sqrt{6}}\right)\right)^2, \tag{6}$$

with x_0 any constant. The Painlevé analysis for (5), put into a travelling frame of reference, exactly determines this particular wave speed $c = \frac{5}{\sqrt{6}}$, which, indeed, is larger than 2.

PAINLEVE ANALYSIS OF EQUATION, $f_{xx} + c f_x - f^2 + f = 0$

SUBSTITUTE $u_0\, g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE [$2\,alfa,\ alfa-2$]

* COEFFICIENT OF $g^{2\,alfa}$ IS $-{u_0}^2$

* COEFFICIENT OF g^{alfa-2} IS $u_0\,(alfa-1)\,alfa$

FOR EXPONENTS ($2\,alfa$) AND ($alfa-2$) OF g, DO

WITH $alfa=-2$, POWER OF g is $-4 \longrightarrow$ SOLVE FOR u_0

TERM $-(u_0-6)\,u_0\,\frac{1}{g^4}$ IS DOMINANT IN EQUATION.

WITH $u_0=6 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\,g^{alfa}+u_r\,g^{r+alfa}$ FOR f IN EQUATION

TERM $(r-6)(r+1)\,u_r\,g^{r-4}$ IS DOMINANT IN EQUATION.

THE ONLY NON-NEGATIVE INTEGRAL ROOT IS [$r=6$]

WITH MAXIMUM RESONANCE $=6 \longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{6} g^{k-2}\,u_k$ FOR f IN EQUATION.

WITH $u_0=6$

* COEFFICIENT OF $\frac{1}{g^3}$ IS $-2\,(6c+5u_1)$

$$u_1=-\frac{6c}{5}$$

* COEFFICIENT OF $\frac{1}{g^2}$ IS $-\frac{6(c^2+50u_2-25)}{25}$

$$u_2=-\frac{(c-5)(c+5)}{50}$$

* COEFFICIENT OF $\frac{1}{g}$ IS $-\frac{6(c^3+250u_3)}{125}$

$$u_3=-\frac{c^3}{250}$$

* COEFFICIENT OF 1 IS $-\frac{7c^4+5000u_4-125}{500}$

$$u_4=-\frac{7c^4-125}{5000}$$

* COEFFICIENT OF g IS $-\frac{79c^5-1375c+75000u_5}{12500}$

$$u_5=-\frac{c(79c^4-1375)}{75000}$$

$*$ COEFFICIENT OF g^2 IS $-\dfrac{c^2(6c^2-25)(6c^2+25)}{6250} = 0$

u_6 IS ARBITRARY !

COMPATIBILITY CONDITION: $-\dfrac{c^2(6c^2-25)(6c^2+25)}{6250} = 0,$

$***$ CONDITION IS NOT SATISFIED .$***$

$***$ CHECK FOR FREE PARAMETERS OR PRESENCE OF u_0 $***$

Example 4: The cylindrical KDV Equation

The *cylindrical Korteweg-de Vries* equation,

$$\frac{f_x}{2t} + f_{xxxx} + 6ff_{xx} + 6(f_x)^2 + f_{tx} = 0, \tag{7}$$

has the Painlevé property. One easily determines the coefficient $\frac{1}{2t}$ in (7), by analyzing a cylindrical KdV equation with arbitrary coefficient $a(t)$ of f_x. Integration of the compatibility condition $a(t)_t + 2a(t)^2 = 0$, gives $a(t) = \frac{1}{2t}$.

PAINLEVE ANALYSIS OF EQUATION,

$$a(t)f_x + f_{xxxx} + 6ff_{xx} + 6(f_x)^2 + f_{tx} = 0$$

SUBSTITUTE $u_0\ g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE [$2\ alfa - 2,\ alfa - 4$]

$*$ COEFFICIENT OF $g^{2\ alfa-2}$ IS $6{u_0}^2\ alfa\ (2alfa-1)\ (g_x)^2$

$*$ COEFFICIENT OF g^{alfa-4} IS $u_0\ (alfa-3)(alfa-2)(alfa-1)\ alfa\ (g_x)^4$

FOR EXPONENTS $(2\ alfa - 2)$ AND $(alfa - 4)$ OF g, DO

WITH $alfa = -2$, POWER OF g is $-6 \longrightarrow$ SOLVE FOR u_0

TERM $60\ u_0\ (g_x)^2\ (2(g_x)^2 + u_0)\ \frac{1}{g^6}$ IS DOMINANT IN EQUATION.

WITH $u_0 = -2(g_x)^2 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\ g^{alfa} + u_r\ g^{r+alfa}$ FOR f IN EQUATION

TERM $(g_x)^4(r-6)(r-5)(r-4)(r+1)u_r g^{r-6}$ IS DOMINANT IN EQUATION.

THE 3 NON-NEGATIVE INTEGRAL ROOTS ARE

$[r = 4,\ r = 5,\ r = 6]$

WITH MAXIMUM RESONANCE = 6 $\longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{6} g^{k-2} u_k$ FOR f IN EQUATION.

WITH $u_0 = -2(g_x)^2$

* COEFFICIENT OF $\frac{1}{g^5}$ IS $120(g_x)^4(2g_{xx} - u_1)$

  $u_1 = 2g_{xx}$

* COEFFICIENT OF $\frac{1}{g^4}$ IS

  $-12(g_x)^2(4g_x g_{xxx} - 3(g_{xx})^2 + 6u_2(g_x)^2 + g_t g_x)$

  $$u_2 = -\frac{4g_x g_{xxx} - 3(g_{xx})^2 + g_t g_x}{6(g_x)^2}$$

* COEFFICIENT OF $\frac{1}{g^3}$ IS

$$4((g_x)^3 a(t) + (g_x)^2 g_{xxxx} - 4g_x g_{xx} g_{xxx} + 3(g_{xx})^2 - g_t g_x g_{xx} - 6u_3(g_x)^4 + g_{tx}(g_x)^2)$$

$$u_3 = ((g_x)^3 a(t) + (g_x)^2 g_{xxxx} - 4g_x g_{xx} g_{xxx} + 3(g_{xx})^2 - g_t g_x g_{xx} + g_{tx}(g_x)^2)/(6(g_x)^4)$$

* COEFFICIENT OF $\frac{1}{g^2}$ IS 0

  u_4 IS ARBITRARY !

  COMPATIBILITY CONDITION IS SATISFIED !

* COEFFICIENT OF $\frac{1}{g}$ IS 0

  u_5 IS ARBITRARY !

  COMPATIBILITY CONDITION IS SATISFIED !

* COEFFICIENT OF 1 IS $\frac{a(t)_t + 2a(t)^2}{6}$

  v_6 IS ARBITRARY !

  COMPATIBILITY CONDITION: $\frac{a(t)_t + 2a(t)^2}{6} = 0$,

*** CONDITION IS NOT SATISFIED .***

*** CHECK FOR FREE PARAMETERS OR PRESENCE OF u_0 ***

Example 5: The Fitzhugh-Nagumo Equation

We recently discovered that the *Fitzhugh-Nagumo* equation,

$$u_t - u_{xx} - u(1-u)(u-a) = 0, \tag{8}$$

has a closed form travelling wave solution, $u(t,x) = U(x-ct)$, where

$$U(\xi) = \left(1 + \exp(-\frac{\xi}{\sqrt{2}})\right)^{-1}, \tag{9}$$

and $c = \frac{2a-1}{\sqrt{2}}$. Carrying out the Painlevé test for (8), in a travelling frame, leads to a compatibility condition which for $u_0 = \sqrt{2}$ factor into

$$c\left(c - \frac{(2a-1)}{\sqrt{2}}\right)\left(c + \frac{(a+1)}{\sqrt{2}}\right)\left(c + \frac{(a-2)}{\sqrt{2}}\right) = 0. \tag{10}$$

So, we see that one of the roots corresponds with the wave speed in (9).

PAINLEVE ANALYSIS OF EQUATION, $f_{xx}+cf_x-f^3+(a+1)f^2-af=0$

SUBSTITUTE $u_0\, g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE $[3alfa,\ 2alfa,\ alfa-2]$

* COEFFICIENT OF g^{3alfa} IS $-u_0{}^3$
* COEFFICIENT OF g^{2alfa} IS $u_0{}^2\,(a+1)$
* COEFFICIENT OF g^{alfa-2} IS $u_0\,(alfa-1)\,alfa$

FOR EXPONENTS $(3alfa)$ AND $(2alfa)$ OF g, $alfa = 0$ IS NON-NEGATIVE.

FOR EXPONENTS $(3alfa)$ AND $(alfa-2)$ OF g, DO

WITH $alfa = -1$, POWER OF g is $-3 \longrightarrow$ SOLVE FOR u_0

TERM $-u_0(u_0{}^2-2)\frac{1}{g^3}$ IS DOMINANT IN EQUATION.

WITH $u_0{}^2 = 2 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\, g^{alfa} + u_r\, g^{r+alfa}$ FOR f IN EQUATION

TERM $(r-4)(r+1)\,u_r\,g^{r-3}$ IS DOMINANT IN EQUATION.

THE ONLY NON-NEGATIVE INTEGRAL ROOT IS $[r=4]$

WITH MAXIMUM RESONANCE = 4 $\longrightarrow$ CHECK RESONANCES.
SUBSTITUTE POWER SERIES $\sum_{k=0}^{4} u_k g^{k-1}$ FOR f IN EQUATION.
WITH $u_0{}^2 = 2$

* COEFFICIENT OF $\frac{1}{g^2}$ IS $-(u_0 c - 2a + 6u_1 - 2)$

$$u_1 = -\frac{u_0 c - 2a - 2}{6}$$

* COEFFICIENT OF $\frac{1}{g}$ IS $-\frac{u_0 c^2 - 2u_0 a^2 + 2u_0 a + 36u_2 - 2u_0}{6}$

$$u_2 = -\frac{u_0(c^2 - 2a^2 + 2a - 2)}{36}$$

* COEFFICIENT OF 1 IS

$$-\frac{2u_0 c^3 - 3u_0 a^2 c + 3u_0 ac - 3u_0 c - 2a^3 + 3a^2 + 3a + 108u_3 - 2}{27}$$

$$u_3 = -\frac{2u_0 c^3 - 3u_0 a^2 c + 3u_0 ac - 3u_0 c - 2a^3 + 3a^2 + 3a - 2}{108}$$

* COEFFICIENT OF g IS

$$-\frac{c(2u_0 c^3 - 3u_0 a^2 c + 3u_0 ac - 3u_0 c - 2a^3 + 3a^2 + 3a - 2)}{27}$$

u_4 IS ARBITRARY !

COMPATIBILITY CONDITION:

$$-\frac{c(2u_0 c^3 - 3u_0 a^2 c + 3u_0 ac - 3u_0 c - 2a^3 + 3a^2 + 3a - 2)}{27} = 0,$$

* * * CONDITION IS NOT SATISFIED. * * *
*** CHECK FOR FREE PARAMETERS OR PRESENCE OF u_0 ***

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